TIME-HARMONIC PROBLEMS FOR CRACKED FUNCTIONALLY GRADED MAGNETO-ELECTRO-ELASTIC COMPOSITES

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Abstract. *In this study we consider functionally graded magneto-electro-elastic materials (MEEM) subjected to anti-plane time-harmonic load. The purpose is to evaluate the dependence of the stress concentration near the crack tips on the frequency of the applied external load. The mathematical model is described by a boundary value problem for a system of partial differential equations. Due to the existence of fundamental solutions the boundary value problem is reduced to a system of integro-differential equations along the crack. The fundamental solutions are derived in a closed form by the Radon transform. For the numerical solution software code in FORTRAN 77 is created and validated. Simulations show the dependence of the stress intensity factors (SIF) on frequency of the incident wave for different types of load, configurations of cracks and different parameters of inhomogeneity.*

KEYWORDS: MEEM, ANTI-PLANE INCIDENT WAVE, RADON TRANSFORM, SIF

1. Introduction

Piezoelectric/ piezomagnetic composites are new smart materials having wide application in modern engineering structures. They possess magneto-electric effect that doesn't exist in piezoelectric or piezomagnetic phase. This new effect was reported for the first time by VanSuchtelen [1] in 1972. Due to different physical and chemical properties of the constituents in the composite a process called delamination may occur. To avoid the abrupt change of the material properties functionally graded materials (FGM) have been created. First practical application of FGM was in Japan, 1984, during the project SPACEPLANE. Researchers required a material that can created 1000 degrees temperature difference in 10mm thickness. Confronted with this challenge they come up with the novel idea of functionally grading [2], [3].The composition of the FGM varies continuously with dimension and thus eliminates the sharp difference of the material properties of the phases in the composites.

Cracks inevitably exist in magneto-electro-elastic materials. Subjected to external load these cracks may extend and cause mechanical failures. Therefore crack analysis plays an important role in construction of smart structures.

In this study we consider FGMEEM with one or two cracks subjected to time-harmonic incident wave and evaluate the dependence of the stress concentration near the crack tips on the incident wave. The mathematical model is described by a boundary value problem for a system of partial differential equations. The boundary value problem is reduced to a non-hypersingular traction boundary integral equation along the crack. This is possible because of the existence of the fundamental solution, which is derived in a closed form by the Radon transform. After discretization of the cracks the resulting algebraic system is solved numerically. The software code in FORTRAN 77 is validated using available examples in literature.

2. Statement of the problem

The MEEM that we study are transversely isotropic. Therefore we consider a rectangular coordinate system $Ox_1x_2x_3$ and assume Ox_3 is the axis of symmetry and polling direction and Ox_1x_2 is the isotropic plane. The solid is subjected to an external anti-plane mechanical, and in-plane electrical and magnetic time-harmonic load with respect to the isotropic plane. We consider mechanical waves with low frequency and assume that electric and magnetic fields are potential. We also assume that the problem is two dimensional i. e. the material properties are the same in all planes parallel to the isotropic plane. In compact notation the constitutive equations can be written in the following way Sladek et al. [4], Soh and Liu [5]:

$$\sigma_{iJ} = C_{iJKl} u_{K,l} \tag{1}$$

where C_{iJKl} is the generalized elasticity tensor defined as follows:

$$C_{i33l} = \begin{cases} c_{44}, i = l \\ 0, i \neq l \end{cases}, C_{i34l} = C_{i43l} = \begin{cases} e_{15}, i = l \\ 0, i \neq l \end{cases},$$

$$C_{i35l} = C_{i53l} = \begin{cases} q_{15}, i = l \\ 0, i \neq l \end{cases}$$

$$C_{i44l} = \begin{cases} -\varepsilon_{11}, i = l \\ 0, i \neq l \end{cases}, C_{i45l} = C_{i54l} = \begin{cases} -d_{11}, i = l \\ 0, i \neq l \end{cases}$$

$$C_{i55l} = \begin{cases} -\mu_{11}, i = l \\ 0, i \neq l \end{cases}.$$

σ_{iJ} is the generalized stress, $\sigma_{iJ} = (\sigma_{i3}, D_i, B_i)$ $i = 1,2$, $J = 3,4,5$ and u_J is the generalized displacement, $u_J = (u_3, \varphi, \psi)$. In this notation c_{44} is the elastic module, e_{15} is the piezoelectric coefficient, q_{15} is the piezomagnetic coefficient, ε_{11} is the dielectric permittivity, μ_{11} is the magnetic permeability, d_{11} is the magnetoelectric coefficient, u_3 is the coordinate of the displacement vector along Ox_3, φ and ψ are electric and magnetic potential respectively, σ_{i3} is the mechanical stress, D_i and B_i are the components of the electric displacement and magnetic induction respectively. Here comma means differentiation and we assume summation under repeated indexes. We use quasi-static approximation of Maxwell equation, because the frequency of the incident wave is low compared with frequency of the electromagnetic wave. The electric current density of free currents $J_f = 0$. We obtain (see Sladek et al [4]):

$$\nabla \times E = 0, \quad \nabla \times H = 0 \tag{2}$$

From (2) we conclude that $E = -grad\varphi$ and $H = -grad\psi$. Using equation of motion, and also $div\, D = 0$ when there is no free charges and $div\, B = 0$ we obtain the governing equations for time harmonic load:

$$\sigma_{iJ,i} + \rho_{JK}\omega^2 u_K = 0 \tag{3}$$

In (3) ω is the frequency of the applied time-harmonic load and $\rho_{JK} = \begin{cases} \rho, J = K = 3 \\ 0, J, K = 4\, or\, 5 \end{cases}$, where ρ is the density.

We suppose that the material properties vary in the following way:

$e_{15}(x) = h(x)e_{15}, c_{44}(x) = h(x)c_{44}, \ q_{15}(x) = h(x)q_{15}$, $\varepsilon_{11}(x) = h(x)\varepsilon_{11}$, $d_{11}(x) = h(x)d_{11}$, $\mu_{11}(x) = h(x)\mu_{11}$, $\rho(x) = h(x)\rho$, where the inhomogeneity function has the following form: $h(x) = e^{2(k_1x_1+k_2x_2)} = e^{2<k,x>}$.

The material is subjected to an incident, anti-plane, time-harmonic wave. When it interacts with the crack a scattered wave is produced. The total displacement and traction in any point of the plane can be found using the superposition principle:

$$u_J = u_J^{in} + u_J^{sc}$$

and

$$t_J = t_J^{in} + t_J^{sc}.$$

Here u_J^{in} and t_J^{in} are the displacement and traction of the incident wave field and u_J^{sc} and t_J^{sc} are the displacement and traction of the scattered by the crack wave field.

The boundary conditions are:

$$t_J^{in} + t_J^{sc}\Big|_{\Gamma} = 0, \qquad (4)$$

Here Γ is the crack, t_J is the total generalized traction defined as $t_J = \sigma_{iJ}n_i$, $n = (n_1, n_2)$ is the normal vector to the crack. This means that the crack is impermeable, i. e. free of mechanical traction, electric charges and currents. We also assume that:

$$u_J(x_1, x_2) = 0 \qquad (5)$$

when $(x_1^2 + x_2^2)^{1/2} \to \infty$.

We will solve the boundary value problem (3),(4),(5) transforming it into an equivalent integro – differential system of equations on the crack and then solve this system numerically.

The fundamental solution of the system of partial differential equations (3) is the solution of:

$$\sigma^*_{iJM,i} + \rho_{JK}\omega^2 u^*_{KM} = -\delta_{JM}\delta(x-\xi), \qquad (6)$$

where $\sigma^*_{iJK} = C_{iJKl}u^*_{KM,l}$, $\delta(x-\xi)$ is the Dirak's delta function and δ_{JM} is the Kronecker's symbol. Following Rangelov et al. [6] and Manolis and Shaw [7] we make the transformation $u^*_{KM} = h^{-1/2}U^*_{KM}$ in (6) and obtain:

$$C_{iJKl}U^*_{KM,ii} + [\rho_{JK}\omega^2 - C_{iJKl}h^{-1/2}(h^{1/2})_{,ii}]U^*_{KM} = -h^{-1/2}\delta_{JM}\delta(x,\xi). \qquad (7)$$

The solution of (7) is found in a closed form by direct and inverse Radon transform and generalized function calculations. Once fundamental solution is derived, following Wang and Zhang [8], Gross et al [9] for piezoelectric case we obtain the following traction boundary integral equation:

$$t_J^{in}(x) = -C_{iJKl}n_i(x)\int_\Gamma [(\sigma^*_{\eta JK}(x,y,\omega)\Delta u_{J,\eta}(y,\omega) - \rho_{QJ}\omega^2 u^*_{QK}(x,y,\omega)\Delta u_J(y,\omega))\delta_{\lambda l} - \sigma^*_{\lambda JK}(x,y,\omega)\Delta u_{J,l}(y,\omega)]n_\lambda(y)d\Gamma(y)$$

where $\Delta u_J = u_J\Big|_{\Gamma^+} - u_J\Big|_{\Gamma^-}$ -the jump of the displacement along the crack or crack opening displacements (COD). The incident wave t_J^{in} is known. The unknown COD are found numerically after crack discretization. If we know COD we can find the scattered field at any point in the solid.

The stress, electric field and magnetic field concentration near crack tips are computed using the expressions:

$$K_{III} = \lim_{x_1 \to \pm c} t_3\sqrt{2\pi(x_1 \mp c)},$$

$$K_E = \lim_{x_1 \to \pm c} E_2\sqrt{2\pi(x_1 \mp c)},$$

$$K_H = \lim_{x_1 \to \pm c} H_2\sqrt{2\pi(x_1 \mp c)},$$

where c is the half-length of the crack.

3. Numerical realization

The crack is divided into 7 boundary elements. The unknown COD are approximated using parabolic shape functions. The material constants for the piezoelectric/piezomagnetic composite $BaTiO_3 / CoFe_2O_4$, that is used in this study can be found in Soh and Liu [5], Li [10]. We compare our results with the results of Zhou and Wang [11] who used dual integral equation method to investigate the dynamic behavior of two parallel equal cracks (see fig. 1). The distance h between the cracks is increasing: $h = 0.2c, ..., 6.5c$, where $c = 5\text{mm}$ is the half length of the crack. The normalized frequency of the incident wave is fixed $\Omega = c\sqrt{\rho c_{44}^{-1}}\omega = 0.4$ and the normalized SIF is $K^*_{III} = \dfrac{K_{III}}{t_3^{in}\sqrt{\pi c}}$.

The comparison is given in fig.3, where the normalized SIF versus the ratio $\dfrac{h}{c}$ is presented.

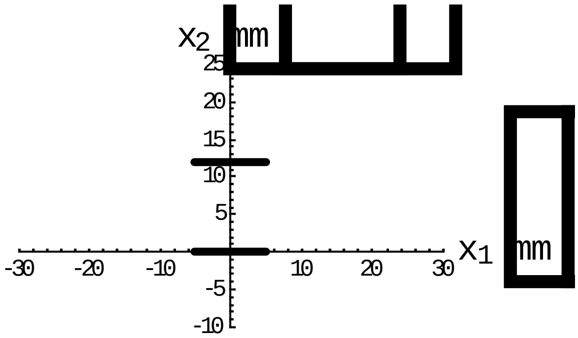


Fig. 1 *Two equal parallel cracks in MEEM.*

We see very good coincidence of the results with difference no more than 2%.

As another example we compare the results for one crack with the results for two collinear equal cracks when the distance between them is large so that the crack interaction is minimal (see fig. 3). The results are obtained by the BIEM. The comparison is given in fig. 4. We see very close results as is to be expected. The maximal difference no more than 3%.

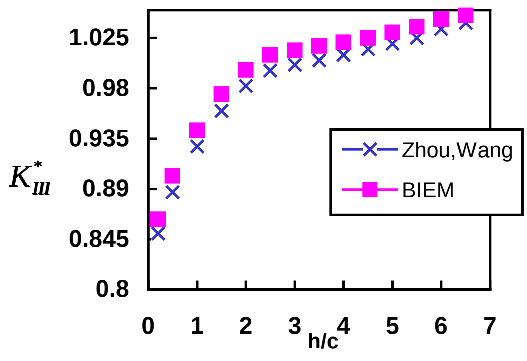


Fig. 2 *Comparison between the results of Zhou and Wang and BIEM for two parallel cracks in MEEM.*

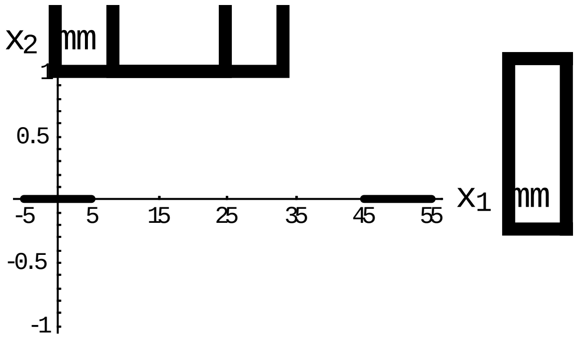


***Fig. 3** Two distant collinear cracks in MEEM.*

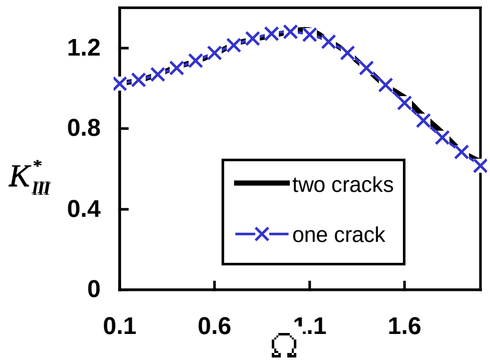


***Fig. 4** Comparison between the results of two distant collinear cracks with results for one crack in MEEM.*

As another validation example we can consider two parallel equal cracks when the distance between them is large. Similar to collinear cracks we expect the crack interaction to be minimal. The results for the normalized SIF with respect to the normalized frequency obtained by the BIEM are given in fig. 5. The comparison shows good coincidence, which is to be expected. The difference is no more than 4%.

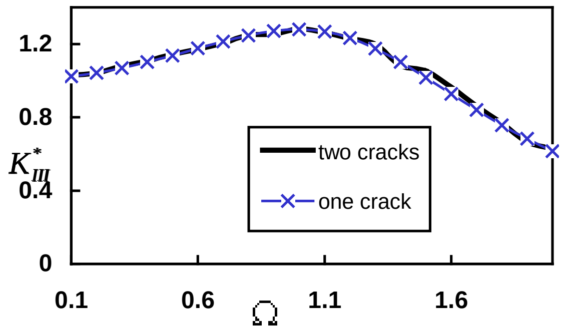
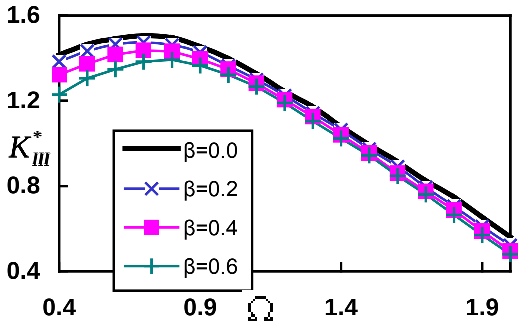


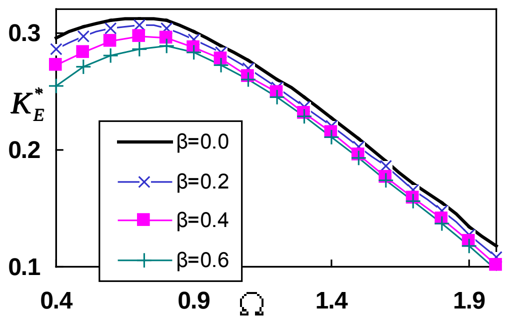
***Fig. 5** Comparison between the results of two distant parallel cracks with results for one crack in MEEM.*

Simulation studies show the dependence of the stress intensity factor, electric field intensity factor (EFIF) and magnetic field intensity factor (MFIF) on the normalized frequency for different parameters of the inhomogeneity function. These parameters are presented in the following way: $(k_1, k_2) = \frac{\beta}{2c}(\sin\alpha, \cos\alpha)$. The normalized EFIF and MFIF are computed as follows: $K_E^* = 10\frac{K_E}{t_3^{in}\sqrt{\pi c}}$ and $K_H^* = 10^4\frac{K_H}{t_3^{in}\sqrt{\pi c}}$.

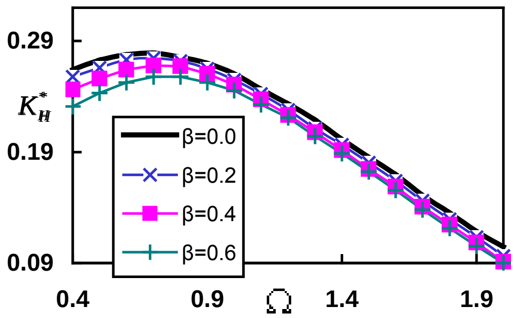
In fig. 6 a), b), c) we present the normalized SIF, EFIF and MFIF versus the normalized frequency for normal incident wave. The cracks are equal and collinear and the distance between them is $h = 0.5c$. The inhomogeneity angle is fixed: $\alpha = 0.0$, while the inhomogeneity magnitude β is increasing: $\beta = 0.0, 0.2, 0.4, 0.6$. The case $\beta = 0$ corresponds to homogeneous material. SIF, EFIF and MFIF are computed at the right crack tip of the left crack. The results show decreasing of the generalized SIF with increasing of the inhomogeneity magnitude β. We also see that this effect is frequency dependent. The conclusion is that the concept of the FGM to reduce stress concentration and improve the strength of the materials works.



6a)



6b)



6c)

***Fig. 6 a),b),c)** SIF, EFIF and MFIF with respect to the normalized frequency for normal incident wave.*

4. Conclusion

In this study we presented numerical solution of a system of integro-differential equations for FGMEEM with one or two cracks, subjected to an incident SH wave. FORTRAN 77 code, based on the BIEM is developed, validated and used in simulations studies. The numerical results show the sensitivity of SIF, EFIF and MFIF to distance between the cracks and parameters of the inhomogeneity function. I also shows that FGM can be used to enhance the strength of the multifunctional materials This software can be further developed to solve problems in FGMEEM with more complex crack configurations, materials with nanoheterogeneities, heterogeneous structures, thermo-elastic problems, MEEM under in-plane waves, problems in finite domains, inverse problems. It also has applications in non-destructive material testing.

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